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Research Article

Further Study on Strong Lagrangian Duality Property for Invex Programs via Penalty Functions

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We apply the quadratic penalization technique to derive strong Lagrangian duality property for an inequality constrained invex program. Our results extend and improve the corresponding results in the literature.

1. Introduction

It is known that Lagrangian duality theory is an important issue in optimization theory and methodology. What is of special interest in Lagrangian duality theory is the so-called strong duality property, that is, there exists no duality gap between the primal problem and its Lagrangian dual problem. More specifically, the optimal value of the primal problem is equal to that of its Lagrangian dual problem. For a constrained convex program, a number of conditions have been obtained for its strong duality property, see, for example, [1–3] and the references therein. It is also well known that penalty method is a very popular method in constrained nonlinear programming [4]. In [5], a quadratic penalization technique was applied to establish strong Lagrangian duality property for an invex program under the assumption that the objective function is coercive. In this paper, we will derive the same results under weaker conditions. So our results improve those of [5].

Consider the following inequality constrained optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in R^n, \quad g_j(x) \le 0, \quad j = 1, \ldots, m, \end{aligned} \tag{P}$$

where $f, g_j (j = 1, \ldots, m) : R^n \to R^1$ are continuously differentiable.

The Lagrangian function for (P) is

$$L(x,\mu) = f(x) + \sum_{j=1}^{m}\mu_j g_j(x), x \in R^n, \quad \mu = (\mu_1, \dots, \mu_m) \in R_+^m. \tag{1.1}$$

The Lagrangian dual function for (P) is

$$h(\mu) = \inf_{x\in R^n} L(x,\mu), \quad \forall \mu \in R_+^m. \tag{1.2}$$

The Lagrangian dual problem for (P) is

$$\sup_{u\in R_+^m} h(\mu). \tag{D}$$

Denote by M_P and M_D the optimal values of (P) and (D), respectively. It is known that weak duality $M_P \geq M_D$ holds. However, there is usually a duality gap, that is, $M_P > M_D$. If $M_P = M_D$, we say that strong Lagrangian duality property holds (or zero duality gap property holds).

Recall that a differentiable function $u : R^n \to R^1$ is invex if there exists a vector-valued function $\eta : R^n \times R^n \to R^n$ such that $u(x) - u(y) \geq \eta^T(x,y)\nabla u(y)$, for all $x, y \in R^n$. Clearly, a differentiable convex function u is invex with $\eta(x,y) = x - y$. It is known from [6] that a differentiable convex function u is invex if and only if each stationary point of u is a global optimal solution of u on R^n.

Let $X \subset R^n$ be nonempty. $u : R^n \to R^1$ is said to be level bounded on X if for any real number t, the set $\{x \in X : u(x) \leq t\}$ is bounded.

It is easily checked that u is level bounded, on X if and only if X is bounded or u is coercive on X if X is unbounded (i.e., $\lim_{x\in X, \|x\|\to+\infty} u(x) = +\infty$).

2. Main Results

In this section, we present the main results of this paper.

Consider the following quadratic penalty function and the corresponding penalty problem for (P):

$$P_k(x) = f(x) + k\sum_{j=1}^{m} g_j^{+2}(x), \quad x \in R^n, \tag{2.1}$$

$$\min_{x\in R^n} P_k(x), \tag{P_k}$$

where the integer $k > 0$ is the penalty parameter.

For any $t \in R^1$, denote that

$$X(t) = \{x \in R^n : g_j(x) \leq t, \ j = 1, \dots, m\}. \tag{2.2}$$

It is obvious that $X(0)$ is the feasible set of (P). In the sequel, we always assume that $X(0) \neq \emptyset$.

We need the following lemma.

Lemma 2.1. *Assume that f is level bounded on $X(0)$, then the solution set of* (P) *is nonempty and compact.*

Proof. It is obvious that problem (P) and the following unconstrained optimization problem have the same optimal value and the same solution set,

$$\min \overline{f}(x), \tag{$\overline{\mathrm{P}}$}$$

where

$$\overline{f}(x) = \begin{cases} f(x), & x \in X_0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

It is obvious that $\overline{f} : R^n \to R^1 \cup \{+\infty\}$ is proper, lower semicontinuous, and level bounded. By [7, Theorem 1.9], the solution set of ($\overline{\mathrm{P}}$) is nonempty and compact. Consequently, the solution set of (P) is nonempty and compact. □

Now we establish the next lemma.

Lemma 2.2. *Suppose that there exists $t_0 > 0$ such that f is level bounded on $X(t_0)$, and there exists $k^* > 0$ and $m_0 \in R^1$ such that*

$$P_{k^*}(x) \geq m_0, \quad \forall x \in R^n. \tag{2.4}$$

Then

(i) *the optimal set of* (P) *is nonempty and compact;*

(ii) *there exists $k^{*'} > 0$ such that for each $k \geq k^{*'}$, the penalty problem (P_k) has an optimal solution x_k; the sequence $\{x_k\}$ is bounded and all of its limiting points are optimal solutions of* (P).

Proof. (i) Since $X(0) \subset X(t_0)$ is nonempty and f is level bounded on $X(t_0)$, we see that f is level bounded on $X(0)$. By Lemma 2.1, we conclude that the solution set of (P) is nonempty and compact.

(ii) Let $x_0 \in X(0)$ and $k^{*\prime} \geq k^* + 1$ satisfy

$$\frac{f(x_0) + 1 - m_0}{k^{*'} - k^*} \leq t_0^2. \tag{2.5}$$

Note that when $k \geq k^{*'}$,

$$P_k(x) = f(x) + k^* \sum_{j=1}^{m} g_j^{+2}(x) + (k - k^*) \sum_{j=1}^{m} g_j^{+2}(x) \geq m_0 + (k - k^*) \sum_{j=1}^{m} g_j^{+2}(x). \tag{2.6}$$

Consequently, $P_k(x)$ is bounded below by m_0 on R^n. For any fixed $k \geq k^* + 1$, suppose that $\{y_l\}$ satisfies $P_k(y_l) \to \inf_{x\in R^n} P_k(x)$. Then, when l is sufficiently large,

$$f(x_0) + 1 = P_k(x_0) + 1 \geq p_k(y_l) = f(y_l) + k\sum_{j=1}^{m} g_j^{+2}(y_l) \geq m_0 + (k - k^*)\sum_{j=1}^{m} g_j^{+2}(y_l). \tag{2.7}$$

Thus,

$$\frac{f(x_0) + 1 - m_0}{k - k^*} \geq \sum_{j=1}^{m} g_j^{+2}(y_l) \geq g_j^{+2}(y_l), \quad j = 1, \ldots, m. \tag{2.8}$$

It follows that

$$g_j^{+}(y_l) \leq \left[\frac{f(x_0) + 1 - m_0}{k - k^*}\right]^{1/2} \leq \left[\frac{f(x_0) + 1 - m_0}{k^{*'} - k^*}\right]^{1/2} \leq t_0, \quad j = 1, \ldots, m. \tag{2.9}$$

That is, $y_l \in X(t_0)$, when l is sufficiently large. From (2.7), we have

$$f(y_l) \leq f(x_0) + 1, \tag{2.10}$$

when l is sufficiently large. By the level boundedness of f on $X(t_0)$, we see that $\{y_l\}$ is bounded. Thus, there exists a subsequence $\{y_{l_p}\}$ of $\{y_l\}$ such that $y_{l_p} \to x_k$ as $p \to +\infty$. Then

$$P_k\left(y_{l_p}\right) \longrightarrow P_k(x_k) = \inf_{x\in R^n} P_k(x). \tag{2.11}$$

Moreover, $x_k \in X(t_0)$. Thus, $\{x_k\}$ is bounded. Let $\{x_{k_i}\}$ be a subsequence which converges to x^*. Then, for any feasible solution x of (P), we have

$$f(x_{k_i}) + k_i\sum_{j=1}^{m} g_j^{+2}(x_{k_i}) \leq f(x). \tag{2.12}$$

That is,

$$m_0 + (k_i - k^*)\sum_{j=1}^{m} g_j^{+2}(xk_i) \leq f(x_{k_i}) + k^*\sum_{j=1}^{m} g_j^{+2}(x_{k_i}) + (k_i - k^*)\sum_{j=1}^{m} g_j^{+2}(x_{k_i}) \leq f(x), \tag{2.13}$$

namely,

$$\sum_{j=1}^{m} g_j^{+2}(x_{k_i}) \leq \frac{f(x) - m_0}{k_i - k^*}. \tag{2.14}$$

Passing to the limit as $i \to +\infty$ and noting that $x_{k_i} \to x^*$, we have

$$\sum_{i=1}^{m} g_j^{+2}(x^*) \leq 0. \tag{2.15}$$

Hence,

$$g_j^+(x^*) = 0, \quad j = 1, \ldots, m. \tag{2.16}$$

It follows that

$$g_j(x^*) \leq 0, \quad j = 1, \ldots, m. \tag{2.17}$$

Consequently, $x^* \in X(0)$. Moreover, from (2.12), we have $f(x_{k_i}) \leq f(x)$. Passing to the limit as $i \to +\infty$, we obtain $f(x^*) \leq f(x)$. By the arbitrariness of $x \in X(0)$, we conclude that x^* is an optimal solution of (P). □

Remark 2.3. If $f(x)$ is bounded below on R^n, then for any $k > 0$, $P_k(x)$ is bounded below on R^n.

The next proposition presents sufficient conditions that guarantee all the conditions of Lemma 2.2.

Proposition 2.4. *Any one of the following conditions ensures the validity of the conditions of Lemma 2.2*

(i) $f(x)$ *is coercive on* R^n*;*

(ii) *the function* $\max\{f(x), g_j^+(x), j = 1, \ldots, m\}$ *is coercive on* R^n *and there exist* $k^* > 0$ *and* $m_0 \in R^1$ *such that*

$$P_{k^*}(x) \geq m_0, \quad \forall x \in R^n. \tag{2.18}$$

Proof. We need only to show that if (ii) holds, then the conditions of Lemma 2.1 hold, since condition (i) is stronger than condition (ii). Let $t_0 > 0$. We need only to show that f is coercive on $X(t_0)$. Otherwise, there exists $\sigma > 0$ and $\{y_k\} \subset X(t_0)$ with $\|y_k\| \to +\infty$ satisfying

$$f(y_k) \leq \sigma. \tag{2.19}$$

From $\{y_k\} \subset X(t_0)$, we deduce

$$g_j(y_k) \leq t_0, \quad j = 1, \ldots, m. \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$\max\Big\{f(y_k), g_j^+(y_k),\ j = 1, \ldots, m\Big\} \leq \max\{\sigma, t_0\}, \tag{2.21}$$

contradicting the coercivity of $\max\{f(x), g_j^+(x),\ j = 1, \ldots, m\}$ since $\|y_k\| \to +\infty$ as $k \to +\infty$. □

The next proposition follows immediately from Lemma 2.2 and Proposition 2.4.

Proposition 2.5. *If one of the two conditions (i) and (ii) of Proposition 2.4 holds, then the conclusions of Lemma 2.2 hold.*

The following theorem can be established similarly to [5, Theorem 4] by using Lemma 2.2.

Theorem 2.6. *Suppose that $f, g_j (j = 1, \ldots, m)$ are all invex with the same η and the conditions of Lemma 2.2 hold, then, $M_P = M_D$.*

Corollary 2.7. *Suppose that $f, g_j (j = 1, \ldots, m)$ are all invex with the same η and one of the conditions (i) and (ii) of Proposition 2.4 holds, then, $M_P = M_D$.*

Example 2.8. Consider the following optimization problem

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & x \in R^1, \quad x^2 \le 0. \end{aligned} \tag{P}$$

It is easy to see that both the objective function and the constraint function are convex and thus invex. Note that the objective function $f(x) = x \to -\infty$ as $x \to -\infty$. It follows that $\lim_{\|x\| \to +\infty} f(x) = +\infty$ does not hold. Consequently, all the results in [5] are not applicable. However, it is easily checked that the conditions of our Corollary 2.7 hold and, hence, $M_P = M_D$.

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